QUANTIZATION IN ADS AND THE ADS/CFT CORRESPONDENCE

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Abstract

The quantization of a scalar field in AdS leads no two kinds of normalizable modes, usually called regular and irregular modes. The regular one is easily taken into account in the standard prescription for the AdS/CFT correspondence. The irregular mode requires a modified prescription which we argue is not completely satisfactory. We discuss an alternative quantization in AdS which incorporates boundary terms in a natural way. Within this quantization scheme we present an improved prescription for the AdS/CFT correspondence which can be applied to both, regular and irregular modes. Boundary conditions other than Dirichlet are naturally treated in this new improved setting.

I. INTRODUCTION

The AdS/CFT correspondence relates a quantum field theory in the bulk of AdS with a conformal field theory living at its border [1]. In the simplest case, that for a scalar field, the quantization in the bulk produces two kinds of normalizable modes depending on the mass of the field [2]. For large masses there is only one mode, the regular one, while for small masses regular and irregular modes can propagate in the bulk. The AdS/CFT correspondence, as originally proposed [1], considered only the conformal theory originated from the regular mode. Subsequently, a modified prescription was presented to take into account the irregular mode as well [3]. When one of them is mapped to the border the other one becomes the source for the conformal operator. However, it is not clear, in both prescriptions, which modes are being taken into account. It seemed that there was no way for selecting which mode is mapped and which one is becoming the source. Then, it was proposed an alternative quantization in AdS which can effectively select which mode is propagating in the bulk [4]. Within this new framework the AdS/CFT correspondence can be improved in the sense that it is possible to keep track of the modes in the bulk. In this new situation we have a precise picture of the physics in the bulk and in the border.

In Section 2 we present the usual quantization in AdS. In the next section we present the AdS/CFT correspondence for regular and irregular modes and discuss its limitations. In Section 4 we present the alternative quantization in AdS. Finally, in Section 5 we show how the AdS/CFT prescription can be improved.

II. QUANTIZATION IN AdS$_{d+1}$

Consider a scalar field ϕ with mass m in AdS. Its energy is defined in the standard way through the canonical energy-momentum tensor $T^{\mu\nu}$. The quantization of this field, with conserved, positive and finite energy, and no energy flux at the border, leads to the regular mode ϕ_R. Near the border, at $\rho = \pi/2$, it behaves as $\phi_R = \epsilon^{\Delta_+}$, where $\epsilon = \cos\rho$ and

$$\Delta_\pm = \frac{d}{2} \pm \nu, \qquad \nu = \sqrt{\frac{d^2}{4} + m^2}. \tag{1}$$

Since the energy-momentum tensor is defined up to improvement terms

$$t_{\mu\nu} = T_{\mu\nu} + \beta(g_{\mu\nu} D^2 - D_\mu D_\nu + R_{\mu\nu})\phi^2, \tag{2}$$

we must look for other solutions whose energy, derived from the improved energy-momentum tensor, is also conserved, positive and finite. It is then found that there is another solution whose asymptotic behavior is $\phi_I = \epsilon^{\Delta_-}$ if $\beta = \Delta_-/(2\Delta_- + 1)$ and the mass is constrained by $0 \leq \nu < 1$. This is the so called irregular mode. For any value of the mass, stability of the solutions also requires that $\nu \geq 0$ which is known as the Breitenlohner-Freedman bound $m^2 \geq -d^2/4$ [2].

We then have the following picture of the physics in the bulk. For $\nu > 1$ only the regular mode can be quantized and the irregular one is a classical background. For $0 \leq \nu < 1$ both modes can be quantized and we have two quantum theories in the bulk.

III. THE ADS/CFT CORRESPONDENCE

In the usual prescription for the AdS/CFT correspondence [1] we compute the partition function with the fields ϕ having fixed values ϕ_0 at the border of AdS. These asymptotic fields are then regarded as sources for the conformal theory operator $\mathcal{O}$ which lives in the AdS border

$$\int_{\phi_0} \mathcal{D}\phi \, e^{-S[\phi]} =< e^{\int d^dx \, \mathcal{O}\phi_0} > . \tag{3}$$

This proposal has been verified in a number of situations [5]. In order to compute the lhs of (3) we consider, in first approximation, only the classical solution so that we have

$$e^{-S[\phi_0]} =< e^{\int d^dx \, \mathcal{O}\phi_0} > . \tag{4}$$

To proceed, it is convenient to use the Euclidean version of AdS and also Poincaré coordinates

$$ds^2 = \frac{1}{x_0^2} \sum_\mu (dx^\mu)^2, \tag{5}$$

so that the border is now at $x_0 = 0$. Since the metric diverges at the border we must be careful in defining how we approach it [6]. Near the border the fields will be denoted by $\phi_\epsilon(\vec{x}) = \phi(\epsilon, \vec{x})$. We then absorb all divergences in $\phi_\epsilon(\vec{x})$ before taking the limit $\epsilon \to 0$. For Dirichlet boundary conditions we find, after using the equation of motion

$$S = -\frac{1}{2} \int d^{d+1}x \, \sqrt{g} \, (\partial\phi\partial\phi + m^2\phi^2) = \frac{1}{2} \int d^dx \, \sqrt{h} \, \phi_\epsilon \partial_n \phi_\epsilon, \tag{6}$$

where h is induced metric at the border and ∂_n is the normal derivative. Then, the action is always reduced to a boundary term when it is on-shell. Inserting the solution of the Klein-Gordon equation we get

$$\begin{aligned} S = &\int d^dx \, d^dy \, \phi_\epsilon(\vec{x})\phi_\epsilon(\vec{y}) \times \\ &\int d^dk \, e^{-i\vec{k}\cdot(\vec{x}-\vec{y})} \epsilon^{-d} \left[\Delta_- + \frac{(k\epsilon)^2}{2(1-\nu)} - \frac{\Gamma(1-\nu)}{2^{2\nu-1}\Gamma(\nu)} (k\epsilon)^{2\nu} + \ldots \right], \end{aligned} \tag{7}$$

where $\ldots$ means high order terms in ϵ. The first two terms are proportional to a delta function and its second derivative, respectively. They give rise to contact terms which are usually disregarded since we are interested only in the terms which do not vanish when $\vec{x} \neq \vec{y}$. Also, the leading term gives $\epsilon^{2\nu-d} = \epsilon^{-2\Delta_-}$ which means that to remove all divergences

$$\phi_\epsilon = \lim_{\epsilon \to 0} \epsilon^{\Delta_-} \phi_0. \tag{8}$$

When the integration in $\vec{k}$ is performed we get

$$S = \int d^dx \, d^dy \, \frac{\phi_0(\vec{x})\phi_0(\vec{y})}{|\vec{x} - \vec{y}|^{2\Delta_+}}. \tag{9}$$

After using (4) we find that the conformal dimension of $\mathcal{O}$ is Δ_+ so that the regular mode was captured by the correspondence. From (8) we learn that the source for it, ϕ_ϵ, behaves near the border as ϵ^{Δ_-} and that is the irregular mode. So, even though no restriction has been assumed for ν we are in the situation $\nu > 1$ since only the regular mode was quantized in the bulk.

To capture the irregular mode Klebanov and Witten [3] proposed that instead of using the action (6) in (3) we should use its Legendre transform but not for the full action (6), only for its leading terms in (7). In this situation, the Legendre transform is then defined as

$$\tilde{S}[\tilde{\phi}_0] = S[\phi_0] + (2\Delta_- - d) \int \frac{d^d k}{(2\pi)^d} \phi_0(\vec{k}) \tilde{\phi}_0(-\vec{k}), \tag{10}$$

and we get

$$\tilde{S}[\tilde{\phi}_0] = \int d^d x \, d^d y \; \frac{\tilde{\phi}_0(\vec{x}) \tilde{\phi}_0(\vec{y})}{|\vec{x} - \vec{y}|^{2\Delta_-}}. \tag{11}$$

Now, using (4) for $\tilde{S}$ we get that the conformal dimension of $\mathcal{O}$ is indeed Δ_-, so that the irregular mode was taking into account. This means that ν is in the range $0 \leq \nu < 1$ since this is the condition for the irregular mode to be quantized in the bulk. Notice, however, that no assumption on ν was made. So, in both cases, the physical picture is quite obscure since we need to impose by hand which mode is propagating in the bulk and which mode remains classical.

Moreover, the Legendre transformation (10) can be carried out for any value of ν, and not only for $0 \leq \nu < 1$, since no restriction on ν was assumed. Also this prescription does not work when $\nu = 0$, that is, when $\Delta_+ = \Delta_-$. In this case we would expect that the action and its Legendre transform would be proportional to each other. However, the leading term in (7) is now $\log k$ and its Legendre transform is not proportional to itself. So we will look for an alternative formulation where we can have more control on the relevant modes of the scalar field.

IV. ALTERNATIVE QUANTIZATION IN AdS$_{d+1}$

As discussed in the previous section, only boundary terms contribute to the correspondence. So we would like to consider a quantization framework were they are naturally taken into account [4]. In the usual quantization, the energy is defined through the energy-momentum tensor. In its computation, boundary terms are disregarded. In fact, the addition of boundary terms to the action corresponds to improving the energy momentum-tensor. A definition of energy, which takes into account all boundary terms, is the one which makes use of the Noether current for time displacements [4]. If we have a conserved Noether current J^μ and a Killing vector for infinitesimal displacements, ξ^μ, we can define the energy as

$$E = \int d^d x \; \sqrt{g} J^\mu \xi_\mu. \tag{12}$$

Conservation of energy implies that

$$\dot{E} = \int d^d x \, \sqrt{g} \, [-D_i(J^i \xi_0) + \xi_i D_0 J^i], \tag{13}$$

so that it is a total derivative for time translations. Again, energy is conserved if there is no flux at the border and this will impose conditions on the solutions of the Klein-Gordon equation [4].

An important fact about AdS has to do with the mass term. Since the AdS curvature is constant the origin of the quadratic term in ϕ is ambiguous. It can have a piece coming from the quadratic term in the potential but can also have another part coming from a non-minimal coupling to the background. We then have to consider the action

$$S_0 = -\frac{1}{2} \int d^d x \, \sqrt{g} \, [\partial\phi\partial\phi + (m^2 + \lambda R)\phi^2], \tag{14}$$

where λ is the non-minimal coupling constant and $R = -d(d+1)$ is the Ricci scalar. This means that ϕ has an effective mass $M^2 = m^2 + \lambda R$. This may seems innocuous but the two quadratic terms couple differently to gravity so that when the metric is varied they give different contributions as we shall see immediately.

When we perform infinitesimal variations of the metric and require that the variation of the action vanishes we find a boundary term, reminiscent from the Gibbons-Hawking term. Then we must add to (14) a boundary contribution

$$S = S_0 + \lambda \int d^d x \, \sqrt{h} K \phi^2, \tag{15}$$

where K is the trace of the extrinsic curvature on the boundary which, in global coordinates, is given by $K = -(d - \cos^2\rho)/\sin\rho$.

Performing now infinitesimal variations of ϕ and using the field equations we find

$$\delta S = - \int d^d x \, \sqrt{h} (\partial_n \phi - 2\lambda K \phi) \delta\phi. \tag{16}$$

For Dirichlet boundary condition this term vanishes but for other boundary conditions it does not. We must then add further boundary terms to the action (15). This has been discussed in detail [4,7] but here we will consider only the Dirichlet case for simplicity. Then, no further boundary terms are required by the variational principle.

The Noether current for infinitesimal translations can now be computed and we find

$$J^\mu = -T^\mu_0 - \lambda[\delta^\mu_0 D_\nu(K n^\nu \phi^2) - K n^\mu \partial_0 \phi^2], \tag{17}$$

where $T^{\mu\nu}$ has now the effective mass M instead of m. We have then improved the energy-momentum tensor in a natural way using the Noether current. The energy is them simply $E = \int d^d x \, \sqrt{g} J^0$.

The quantization now proceeds along familiar lines. Requiring that the energy be conserved, positive and finite leads to the two modes with asymptotic behavior $\phi_R = \epsilon^{\Delta_+}$ and $\phi_I = \epsilon^{\Delta_-}$, where now

$$\Delta_\pm = \frac{d}{2} \pm \nu, \qquad \nu = \sqrt{\frac{d^2}{4} + M^2}. \tag{18}$$

For the regular mode we find no constraints. For the irregular one we again find that $0 \leq \nu < 1$ and $\lambda = \Delta_-/2d$. There are two solutions for this equation $\lambda = [d - 1 \pm \sqrt{(d-1)^2 + 16m^2}]/(8d)$. We also found that the Breitenlohner-Freedman bound is still required but now with m replaced by the effective mass M.

This same analysis can be performed for other boundary conditions with similar conclusions [4].

V. IMPROVED ADS/CFT PRESCRIPTION

As discussed before we will consider an improved AdS/CFT prescription which holds for any boundary condition. It reads

$$e^{-S[A_0]} =< e^{\int d^dx\ \mathcal{O} A_0} >, \tag{19}$$

where A_0 is the combination of fields which is fixed at the boundary. For the Dirichlet case it is simply ϕ_0.

We now compute the action (15) for the solution of the Klein-Gordon equation with Dirichlet boundary condition

$$S = \frac{1}{2} \int d^dx\ \sqrt{h} \phi_\epsilon (\partial_n \phi_\epsilon - 2\lambda K_\epsilon \phi_\epsilon). \tag{20}$$

Since there is a boundary term in (15) we now find that

$$\begin{aligned} S =& \int d^dx\, d^dy\ \phi_\epsilon(\vec{x})\phi_\epsilon(\vec{y}) \times \\ & \int d^dk\ e^{-i\vec{k}\cdot(\vec{x}-\vec{y})} \epsilon^{-d} \left[\Delta_- - 2\lambda d + \frac{(k\epsilon)^2}{2(1-\nu)} - \frac{\Gamma(1-\nu)}{2^{2\nu-1}\Gamma(\nu)} (k\epsilon)^{2\nu} + \ldots \right]. \end{aligned} \tag{21}$$

As for the Legendre transformed action we now consider the full action (21) and not just its leading term. We then find

$$\begin{aligned} \tilde{S} =& \int d^dx\, d^dy\ \tilde{\phi}_\epsilon(\vec{x})\tilde{\phi}_\epsilon(\vec{y}) \times \\ & \int d^dk\ e^{-i\vec{k}\cdot(\vec{x}-\vec{y})} \epsilon^{-d} \left[\Delta_- - 2\lambda d + \frac{(k\epsilon)^2}{2(1-\nu)} - \frac{\Gamma(1-\nu)}{2^{2\nu-1}\Gamma(\nu)} (k\epsilon)^{2\nu} + \ldots \right]^{-1}. \end{aligned} \tag{22}$$

Let us consider the case where only regular modes propagate, that is when $\lambda \neq \Delta_-/(2d)$. From (21) we find the same result as in the usual case (9). For the Legendre transformed action $\tilde{S}$ we have to invert the term in square brackets and when that is done it reproduces the same ϵ structure as that in (21). We then get $\tilde{S} = 1/(\Delta_- - 2\lambda d)^2 S$ so that it is also capturing the regular mode in AdS. So both, the action an its Legendre transform, give a boundary theory with conformal dimension Δ_+. This is consistent with the condition $\lambda \neq \Delta_-/(2d)$ which does not allow irregular modes to be quantized in the bulk. We can further check that

$$\phi_\epsilon = \epsilon^{\Delta_-} \phi_0, \qquad \tilde{\phi}_\epsilon = \epsilon^{\Delta_-} \tilde{\phi}_0, \tag{23}$$

showing that the irregular modes are the classical sources for $\mathcal{O}$.

Now we can consider the case when regular and irregular modes propagate in the bulk, that is, $\lambda = \Delta_-/(2d)$. Note that now there are no contact terms in (21). From the action (21) we find that the conformal dimension is Δ_+, so that it is associated with the regular mode propagation in the bulk. Now the Legendre transformed action (22) has a different expansion and we get another ϵ structure. We then find that the conformal dimension is Δ_-, so that it is capturing the irregular mode in the bulk. We can also check that

$$\phi_\epsilon = \epsilon^{\Delta_-}\phi_0, \qquad \tilde{\phi}_\epsilon = \epsilon^{\Delta_+}\tilde{\phi}_0, \tag{24}$$

in agreement with the expected result. It is worth to remark that the absence of contact terms in this case implies that no counterterms are required to remove them.

We can also consider the case with $\nu = 0$ without problems. We find that with this prescription the action and its Legendre transform are indeed proportional and the fields have the correct asymptotic behavior.

VI. CONCLUSIONS

The natural ambiguity for the mass term in the bulk of AdS required that we distinguished between contributions from the potential and from a non-minimal coupling to gravity. The quantization, where use is made of the Killing vector for time translation to define the energy, leads to constraints on the non-minimal coupling. These constraints are associated to the modes which are able to propagate in the bulk. In the AdS/CFT correspondence these constraints tell us which mode is propagating in the bulk so that we have a precise picture of which modes are classical and which ones are quantum. We also improved the correspondence applying the Legendre transform of the full action and not just to its leading terms. In this way a consistent picture emerges where all values of the mass and all boundary conditions can be considered.

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